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Designing planar cubic B-spline curves with monotonic curvature for curve interpolation

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This paper analyzes the curvature distributions of cubic B-spline curves. Firstly, a sufficient geometric condition is presented to guarantee that the curvature of a B-spline curve is monotonic. Then, a design algorithm is developed for cubic B-spline curves with monotonic curvature, as well as an algorithm for curve interpolation with monotonic curvature. Finally, examples are given to demonstrate the effectiveness and efficiency of the new design approach. Compared to Euler curves, the proposed algorithm is easier to compute; it also has an exact solution with boundary conditions. Furthermore, it is compatible with existing commercial CAD software.

1 Introduction

Monotonic curvature plays an important role in designing curves with aesthetic shapes, as in automobile or aircraft design. Industrial design and styling requires aesthetic shapes with monotonic curvature [1]. In conventional parametric computer aided design (CAD) and computer aided manufacturing (CAM) systems, general B-splines are not adequate for aesthetic requirements. Except for a straight line or a circle, monotonic curvature distribution, associated with pleasing shape, is very difficult to achieve.

Curvature plays an important role as a shape descriptor. Farin suggested that a fair curve has a curvature plot with relatively few regions of monotonically varying curvature [2]. Starting from this basis, work on B-spline fairing was developed mainly in three direction: knot-removal-reinsertion methods, optimization methods based on minimizing an energy function, and filtering approaches based on B-spline wavelets. Curve or surface fairing plays an important role in CAD/CAM applications.

Visual curve completion (interpolating a curve segment) is a fundamental problem for human visual understanding [5]. Aesthetically pleasingly shaped curves usually have monotonically varying curvature [2]. While the shape of a curve is primarily defined by its curvature distribution, monotonicity of curvature is not easily achieved and controlled. To overcome this problem, we present a construction approach for B-spline curves, which guarantees monotonic curvature by enforcing simple geometric constraints on the control vectors.

Euler curves have the useful property that their curvature changes linearly with arc length, so are widely used in modeling and shape interpolation [3, 4, 6]. However, Euler curves also have disadvantages for curve interpolation. Firstly, they are defined by transcendental functions, requiring complex mathematical expressions which are very difficult to compute. Secondly, given two endpoints with associated tangents, there is no exact solution for an Euler curve with these boundary conditions. Thirdly, Euler curves are not compatible with current CAD software systems, which are based on NURBS. In order to overcome these drawbacks of Euler curves, we have developed a new interpolation algorithm

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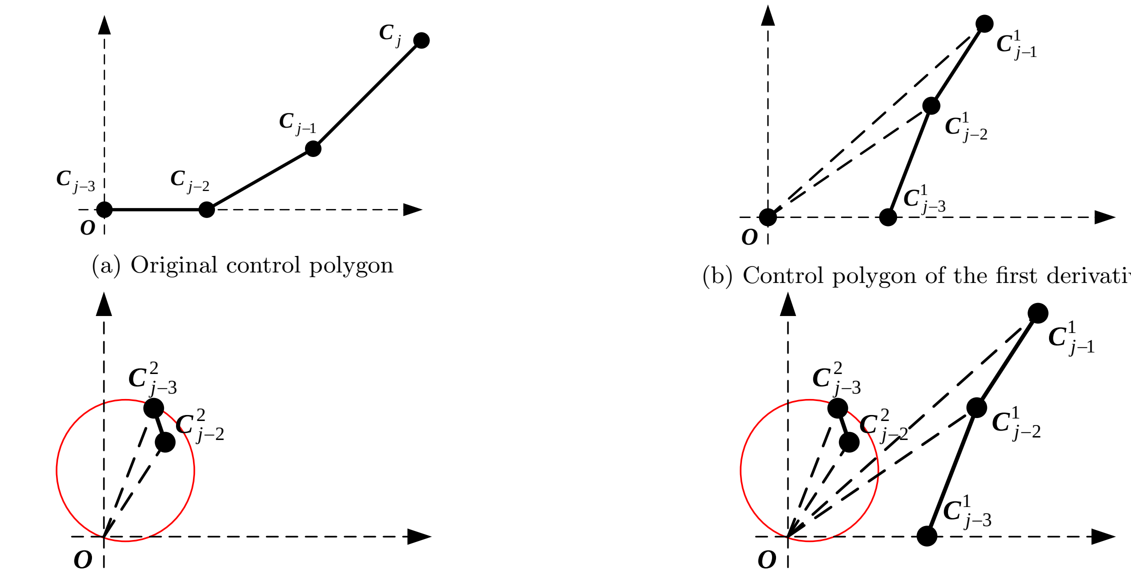


(a) Original control polygon

(b) Control polygon of the first derivative

(c) Control polygon of the second derivative

(d) $\angle\langle P'(t), P''(t)\rangle$

Fig. 1 Control polygons of a cubic B-spline curve and its derivatives

for cubic B-spline curves. The advantages of our method are simple computation, exact interpolation and compatibility with existing CAD systems.

The remainder of this paper is organized as follows. Section 2 gives a sufficient condition for a B-spline curve to have monotonic curvature, and proposes a new approach to design B-spline curves with monotonic curvature variation (MCV). Section 3 provides several examples using this new design approach, demonstrating that curves so constructed not only have aesthetically pleasing shape, but also good monotonic curvature distribution. Finally, Section 4 draws conclusions.

2 Planar cubic B-spline curves with monotonic curvature

2.1 Theory

In this paper, we focus on cubic curves. A planar B-spline curve of degree three is defined by

$$\begin{aligned} P(t) &= \sum_{i=0}^{n} C_i N_{i,3}(t) \\ &= \sum_{i=j-3}^{j} C_i N_{i,3}(t) \quad t \in [t_j, t_{j+1}] \subset [t_3, t_{n+1}] \end{aligned} \tag{1}$$

where C_i are control points (see Fig. 1(a)), and $N_{i,3}(t)$ are B-spline basis functions defined on the knot vector $t = [t_0, \ldots, t_{n+3}]$. The derivatives of this curve can be expressed as

$$\begin{aligned} P'(t) &= \sum_{i=0}^{n-1} C_i^1 N_{i,2}(t) \\ &= \sum_{i=j-2}^{j} C_i^1 N_{i,2}(t) \quad t \in [t_j, t_{j+1}] \subset [t_3, t_{n+1}] \end{aligned} \tag{2}$$

$$\begin{aligned} C_i^1 &= \frac{3-1+1}{u_{i+3+1} - u_{i+1}}(C_{i+1} - C_i) \\ &= \frac{3}{u_{i+4} - u_{i+1}}(C_{i+1} - C_i) \end{aligned} \tag{3}$$

where $P'(t)$ is the first derivative of the B-spline curve, and C_i^1 represents the corresponding control point (see Fig. 1(b)). Proceeding further, we obtain

$$\begin{aligned} P''(t) &= \sum_{i=0}^{n-2} C_i^2 N_{i,1}(t) \\ &= \sum_{i=j-1}^{j} C_i^2 N_{i,1}(t), \quad t \in [t_j, t_{j+1}] \subset [t_3, t_{n+1}] \end{aligned} \tag{4}$$

$$\begin{aligned} C_i^2 &= \frac{3-2+1}{u_{i+3+1} - u_{i+2}}(C_{i+1}^1 - C_i^1) \\ &= \frac{2}{u_{i+4} - u_{i+2}}(C_{i+1}^1 - C_i^1) \end{aligned} \tag{5}$$

where $P''(t)$ is the second derivative, and C_i^2 represents the corresponding control point (see Fig. 1(c)).

Now consider a B-spline curve segment with four control points. It can be represented as $P(t) = \sum_{i=j-3}^{j} C_i N_{i,3}(t)$, $t \in [t_j, t_{j+1}]$. We construct a local coordinate system. Specifically, let the direction of the first control vector denote be the x-axis, and C_{j-3} be the origin point (see Fig. 2).

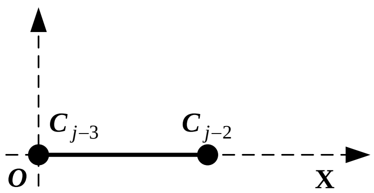


Fig. 2 Local coordinate system of the curve segment

Next, we prove that if the following three geometrical Lemmas are satisfied, the B-spline curve segment on $[t_j, t_{j+1}]$ has monotonic curvature.

Lemma 1 If the first and second derivative control points $\{C^1_{j-3}, C^1_{j-2}, C^1_{j-1}\}$ and $\{C^2_{j-3}, C^2_{j-2}\}$ are located in the same quadrant (see Fig. 1(b,c)), then $\|P'(t)\|' \geq 0$ for $t \in [t_j, t_{j+1}]$.

Proof. From $\|P'(t)\| = (P'(t)(P'(t))^{\mathrm{T}})^{1/2}$, we can derive

$$\|P'(t)\|' = \frac{P'(t)(P''(t))^{\mathrm{T}}}{\|P'(t)\|} \tag{6}$$

Thus, if $P'(t)$ and $P''(t)$ are in the same quadrant, we have that $\|P'(t)\|' \geq 0$ for $t \in [t_j, t_{j+1}]$. □

Lemma 2 If angle $\angle OC^2_{j-2}C^2_{j-3} \geq \pi/2$ (see Fig. 1(c)), then $\|P''(t)\|' \leq 0$ for $t \in [t_j, t_{j+1}]$.

Proof. Because $\angle OC^2_{j-2}C^2_{j-3} \geq \pi/2$, it can be inferred that $\|OC^2_{j-3}\| \geq \|OC^2_{j-2}\|$. As

$$\|P''(t)\| = \frac{t_{j+1} - t}{t_{j+1} - t_j}\|OC^2_{j-3}\| + \frac{t - t_j}{t_{j+1} - t_j}\|OC^2_{j-2}\|,$$

we have $\|P''(t + \Delta t)\| \leq \|P''(t)\|$, where $\Delta t \geq 0$ and $t \in [t_j, t_{j+1}]$. Thus, $\|P''(t)\|' \leq 0$ for $t \in [t_j, t_{j+1}]$. □

Lemma 3 If $k_{\overline{OC^1_{j-2}}} \leq k_{\overline{C^1_{j-2}C^1_{j-1}}} \leq k_{\overline{C^1_{j-3}C^1_{j-2}}}$, where k denotes slope (see Fig. 1(d)), then $\sin(\langle P'(t), P''(t)\rangle) \leq 0$ for $t \in [t_j, t_{j+1}]$, where $\langle\rangle$ denotes the angle between two vectors.

Proof. As the vector $C^1_{j-2}C^1_{j-1} = OC^2_{j-2}$ and $C^1_{j-3}C^1_{j-2} = OC^2_{j-3}$, we have $k_{\overline{OC^1_{j-2}}} \leq k_{\overline{OC^2_{j-2}}} \leq k_{\overline{OC^2_{j-3}}}$. If $k_{\overline{OC^1_{j-2}}} \leq k_{\overline{C^1_{j-2}C^1_{j-1}}} \leq k_{\overline{C^1_{j-3}C^1_{j-2}}}$, from the variation diminishing property, we can derive that $0 \leq \angle\langle P'(t + \Delta t), P''(t + \Delta t)\rangle \leq \angle\langle P'(t), P''(t)\rangle \leq \pi/2$, where $\Delta t \geq 0$, $t \in [t_j, t_{j+1}]$. Then, we have $\sin\langle P'(t+\Delta t), P''(t+\Delta t)\rangle \leq \sin(\langle P'(t), P''(t)\rangle)$. Thus, $\sin(\langle P'(t), P''(t)\rangle)$ is decreasing for $t \in [t_j, t_{j+1}]$. □

Theorem 1. If a B-spline control polygon satisfies Lemmas 1–3, then the curvature of the curve segment is monotone.

Proof. If Lemmas 1–3 are satisfied, then (i) $\|P'(t)\|' \geq 0$, (ii) $\|P''(t)\|' \leq 0$, and (iii) $\sin(\langle P'(t), P''(t)\rangle)' \leq 0$ for $t \in [t_j, t_{j+1}]$. Then, we can conclude that $\kappa(t + \Delta t) \leq \kappa(t)$ as

$$\kappa(t) = \frac{\|P''(t)\| \sin\langle P'(t)P''(t)\rangle}{\|P'(T)\|^2},$$

where $\Delta t \geq 0$. Therefore, $\kappa(t)$ is a decreasing function for $t \in [t_j, t_{j+1}]$, and this curve segment has monotonic curvature. □

Theorem 2. If (i) C^2_{j-2} is located in a fair location (defined in Fig. 3), and (ii) $k_{\overline{OC^1_{j-2}}} \leq k_{\overline{OC^2_{j-3}}}$; (iii) $\{C^1_{j-3}, C^1_{j-2}, C^1_{j-1}\}$ and $\{C^2_{j-3}, C^2_{j-2}\}$ are located in the same quadrant, then the curvature of the associated B-spline curve segment on $[t_j, t_{j+1}]$ is monotone.

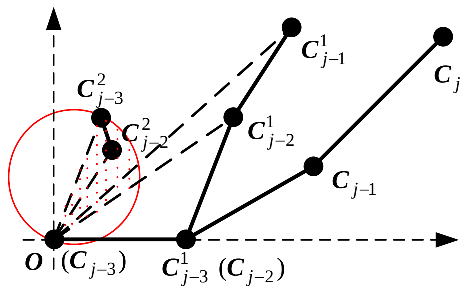


Fig. 3 Fair location. C^2_{j-2} is in a fair location if it lies within the red dashed region, which is bounded by OC^2_{j-3} , OC^1_{j-2} and the red circle with diameter OC^2_{j-3}.

Proof. As the given geometric conditions are satisfied, we have (i) $\{C^1_{j-3}, C^1_{j-2}, C^1_{j-1}\}$ and $\{C^2_{j-3}, C^2_{j-2}\}$ are located in the same quadrant, (2) angle $\angle OC^2_{j-2}C^2_{j-3} \geq \pi/2$, because C^2_{j-2} is located within the red circle with diameter OC^2_{j-3}, and (iii) $k_{\overline{OC^1_{j-2}}} \leq k_{\overline{OC^2_{j-2}}} = k_{\overline{C^1_{j-2}C^1_{j-1}}} \leq k_{\overline{OC^2_{j-3}}} = k_{\overline{C^1_{j-3}C^1_{j-2}}}$ (see Fig. 3). Thus, Theorem 1 is satisfied by the B-spline curve segment, proving the result. □

From Theorem 2, if $\|C^1_{j-2}C^1_{j-1}\| \leq \|C^1_{j-3}C^1_{j-2}\|$ and $C^1_{j-2}C^1_{j-1} \parallel C^1_{j-3}C^1_{j-2}$ are enforced (see Fig. 4), the control polygon satisfies the three geometric conditions. Thus, the corresponding B-spline curve segment has monotonic curvature.

2.2 Designing cubic B-spline curves with monotonic curvature

From Theorem 2, our design approach may be summarized in the following steps.

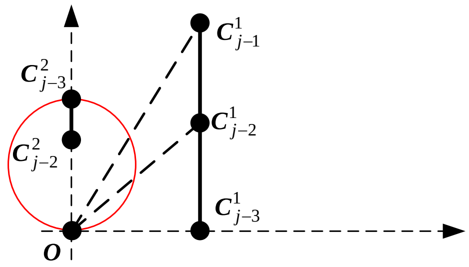


Fig. 4 A special case of Theorem 2

1. Fix the first B-spline control vector. Let $\mathbf{V_0} = C_1 - C_0$ (see Fig. 5);

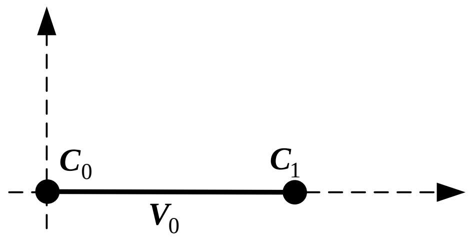


Fig. 5 Determine the initial B-spline control vector

2. Fix the second vector. Let $\mathbf{V_1} = C_2 - C_1$, where $\mathbf{V_0}$ and $\Delta\mathbf{V_0} = \mathbf{V_1} - \mathbf{V_0}$ are located in the same quadrant (see Fig. 6);

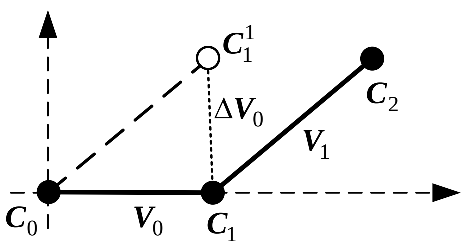


Fig. 6 Determine the second control vector

3. Fix the third vector. Let $\mathbf{V_2} = C_3 - C_2$, where $k_{\overline{OC^1_1}} \leq k_{\overline{OC^2_0}}$, and $\Delta\mathbf{V_1} = \mathbf{V_2} - \mathbf{V_1}$ (C^2_1) is located in a fair location (see Fig. 7).

4. Fix the i-th vector. For a B-spline, if $\mathbf{V_{i-2}}$ and $\mathbf{V_{i-1}}$ are given, $\mathbf{V_i}$ can be determined by steps (1)–(3).

Our design algorithm for a B-spline curve segment satisfies Lemmas 1–3 and hence of Theorem 1: (i) $\{C^1_{j-3}, C^1_{j-2}, C^1_{j-1}\}$ and $\{C^2_{j-3}, C^2_{j-2}\}$ are located in the same quadrant, (ii) $k_{\overline{OC^1_{j-2}}} \leq k_{\overline{OC^2_{j-2}}} \leq k_{\overline{OC^2_{j-3}}}$, and (iii) $\angle OC^2_{j-2}C^2_{j-3} \geq \pi/2 (j = 3, \ldots, n)$. Therefore, the curvature of the designed curve is monotone.

Pseudocode is given in Algorithm 1.

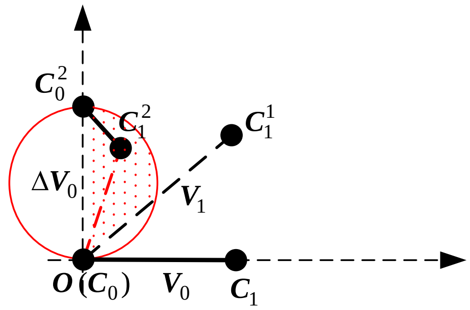


Fig. 7 Determine the third control vector

Algorithm 1

Input: number of control edges n, knot vector t and initial conditions.
Output: planar cubic B-spline curve with monotonic curvature.
Initialize: $\mathbf{C_0} = (0, 0, 0)$,$\mathbf{C_1} = (a, 0, 0)$ and $\mathbf{V_0} = C_1 - C_0$, where $a > 0$
input $\Delta\mathbf{V_0} = \mathbf{V_1} - \mathbf{V_0}$ lying in the first quadrant
for $i = 1$ **to** $n - 1$ **do**
  input $\Delta\mathbf{V_i} = \mathbf{V_{i+1}} - \mathbf{V_i}$ in a fair location
end for

2.3 Curve interpolation with B-spline curves with monotonic curvature

We now consider curve interpolation given two specified endpoints with associated tangent directions: we wish to find an appropriate curve satisfying these G^1 conditions. Algorithm 2 uses a knot sequence which is uniformly spaced everywhere except at its ends. Due to affine invariance, without loss of generality we can fix the starting control point C_0 at the origin, and let the tangent direction T_0 be the x axis (see Fig. 8). The end control point should be located in the first quadrant with a proper tangent direction, limited between the two dashed lines (see Fig. 8).

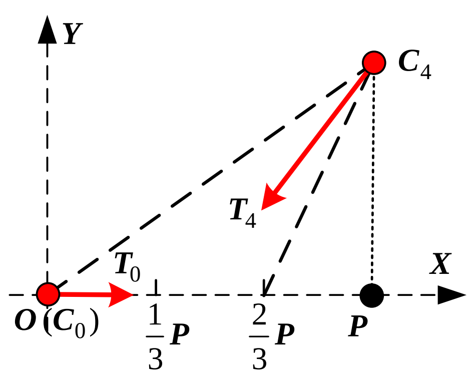


Fig. 8 Given data to interpolate

Compute the intersection M of the two tangent directions. Select parameters t_1, t_{2x}, t_{2y} and t_3 for the middle three control points respectively. Set $C_1 = t_1 P$, $C_3 = (1 - t_3)M + t_3 C_4$, $C_2 = (t_{2x} C_4.x, t_{2y} C_4.y)$ (see Fig. 9). From Eq. (3), Eq. (5) and Lemma 1, we can deduce that $0 < t_1 < 1/6$, $0 < t_3 < 4/5$, $3t_1 < t_{2x} < C_3.x/C_4.x$, $0 < t_{2y} < 1/2t_3$. If the intersection point M is located between the two dashed lines in Fig. 8, this can always be done.

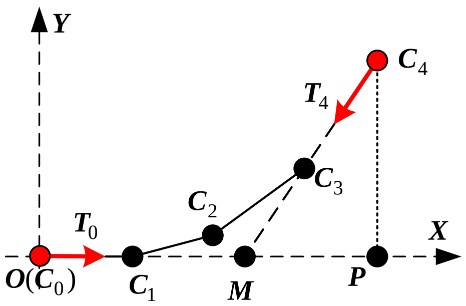


Fig. 9 Generating the middle control points

Pseudocodes for the procedure is given in Algorithm 2.

Algorithm 2

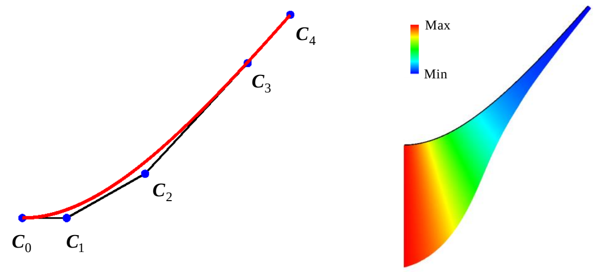
Input: Two endpoints with associated tangent directions (see Fig. 8)
Output: Planar cubic B-spline segment with monotonic curvature.
Initialize: $\delta = 0.02, n = 4$

```
for t1 = 0 to 1/6 step δ do
  C1 = t1 P
  for t3 = 0 to 4/5 step δ do
    C3 = (1 − t3)M + t3 C4
    for t2x = 3t1 to C3.x/C4.x step δ do
      C2.x = t2x C4.x
      for t2y = 0 to t3/2 step δ do
        Success = False
        C2.y = t2y C4.y
        C2 = (C2.x, C2.y)
        Compute Ci^1, Ci^2
        if (Ci^1 and Ci^2 satisfy Lemmas 1–3) then
          return control points
        end if
      end for
    end for
  end for
end for
```

3 Examples of curve interpolation

This section gives several examples are given to demonstrate the effectiveness of the new design approach of Alg. 1 and Alg. 2.

The first example is a cubic B-spline curve with five control points (see Fig. 10(a)). From Theorem 1, it can be inferred that the B-spline curve has monotonic curvature as shown in Fig. 10(b).

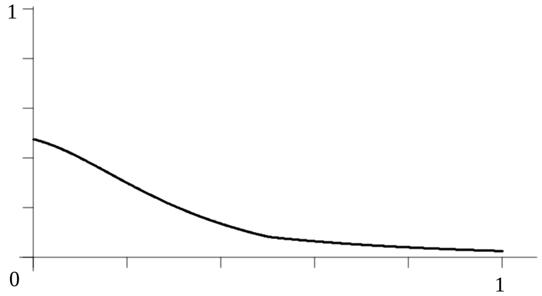


(a) The generated B-spline curve (b) Its curvature comb

Fig. 10 Designing a B-spline curve using Alg. 1

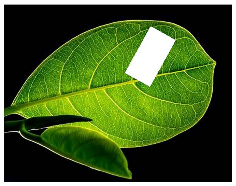
Figures 11–13 show applications of Alg. 2 to curve completion in occluded objects. Fig. 11(a) is a partially occluded bird, and in Fig. 11(b) we apply a cubic B-spline curve constructed by our method to complete its boundary. Fig. 12 shows an example of leaf vein completion. Lastly, the curve in Fig. 13 completes the edge of a wine glass in an artistic photograph.

(a) Partially occluded bird (b) Boundary curve completion

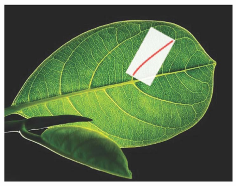


(c) Corresponding curvature plot

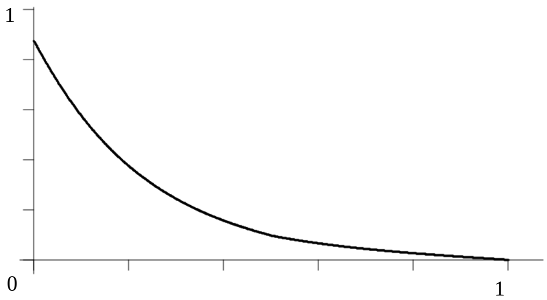
Fig. 11 Bird silhouette completion using Alg. 2



(a) Partially occluded leaf



(b) Leaf vein completion

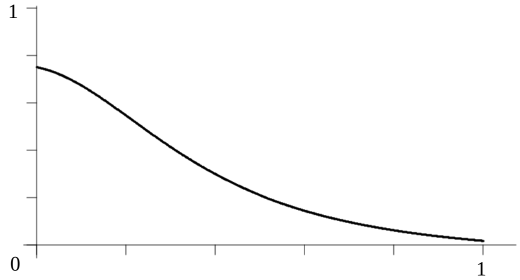


(c) Corresponding curvature plot

Fig. 12 Leaf vein completion using Alg. 2

(a) Partially occluded wine glass

(b) Silhouette completion



(c) Corresponding curvature plot

Fig. 13 Wine glass silhouette completion using Alg. 2

This paper describes a novel approach for designing B-spline curves with monotonic curvature, which can also be used for curve interpolation. Firstly three sufficient conditions for monotonicity curvature of B-spline curves are proved. Then, a new sufficient geometrical condition is given. Based on these conditions, two algorithms for generating B-spline curves with monotonic curvature are also developed. We also give an algorithm to construct a B-spline curve with monotonic curvature which interpolates two given point and tangent direction pairs. Finally, several examples of curve completion using this method are demonstrated, with curvature plots showing their monotonic curvature distributions. Like Euler curves, the curves from our approach have several nice aesthetic properties. However, as B-spline curves, they are simple to compute and interrogate. They are also easier to use in conjunction with existing CAD software.

Acknowledgements

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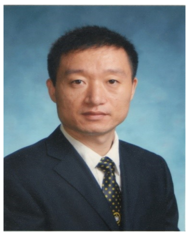

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